PROBLEMS FOR PERSONAL CONTEST

CHOOSE ANY 3 OUT OF 5

Problem 1. Show that if

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

is a companion matrix with distinct eigenvalues $\lambda_1, ..., \lambda_n$, then

$$VCV^{-1} = \text{diag}(\lambda_1, ..., \lambda_n)$$

where

$$V = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

Problem 2. Prove the following error estimate for the Simpson's rule:

$$\int_a^b f(x)\,dx - \frac{b-a}{6}\Big(f(a) + 4f(\frac{a+b}{2}) + f(b)\Big) = -\Big(\frac{b-a}{2}\Big)^5 \frac{f^{(4)}(\xi)}{90}$$

where $a < \xi < b$.

Problem 3. Consider the fixed point iteration method to solve non-linear equation $f(x) = 0$

$$x_{n+1} = g(x_n). \tag{1}$$

a. State the necessary conditions for existence and uniqueness of a fixed-point $x = \alpha$ in (1), and derive the criteria that determines the order of convergence.

b. Consider instead the fixed-point iteration

$$x_{n+1} = G(x_n) = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n}.$$

Show that if α is a fixed-point of $g(x)$, then it is also a fixed point of $G(x)$.

Problem 4. Consider the following parabolic equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

where $a > 0$ is a constant. Consider the following finite difference scheme

$$\frac{1}{12}\frac{v_{m+1}^{n+1} - v_{m+1}^{n}}{k} + \frac{5}{6}\frac{v_{m}^{n+1} - v_{m}^{n}}{k} + \frac{1}{12}\frac{v_{m-1}^{n+1} - v_{m-1}^{n}}{k}$$
$$= a\frac{(\delta^2 v)_m^{n+1} + (\delta^2 v)_m^n}{2}$$

where δ^2 is the standard central difference operator in space, namely,

$$(\delta^2 v)_m^n = \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

and $k, h > 0$ are the time step size and the mesh size respectively. It is known that the scheme is second order in time and fourth order in space.
How to modify the above scheme for the equation

$$\frac{\partial u}{\partial t} = \sigma(x)\frac{\partial^2 u}{\partial x^2}$$

where $\sigma(x) \geq \sigma_0 > 0$, so that the resulting scheme is still second order in time and fourth order in space?

Problem 5. Let A and B be $n \times n$ matrices, A non-singular. Consider solving the linear system of equations

$$\begin{aligned} A\,\mathbf{x}_1 + B\,\mathbf{x}_2 &= \mathbf{b}_1 \\ B\,\mathbf{x}_1 + A\,\mathbf{x}_2 &= \mathbf{b}_2 \end{aligned}$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^n$. Find the necessary and sufficient condition(s) for the convergence of the iterative method

$$\begin{aligned} A\,\mathbf{x}_1^{k+1} &= \mathbf{b}_1 - B\,\mathbf{x}_2^k \\ A\,\mathbf{x}_2^{k+1} &= \mathbf{b}_2 - B\,\mathbf{x}_1^k \end{aligned}$$

for some initial guess $\mathbf{x}_1^0$ and $\mathbf{x}_2^0$.